# On the Loading Surface of Microinhomogeneous Materials

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**Abstract.** The main idea of the report is that the yield surface can be presented as thick, "foamed surface" with dimensionality exceeding two. By other words, perhaps, the yield surface is fractal, i.e. it belongs to the geometric objects with fractional dimensionality. Apparently, fractal structure of the yield surface is determined by the fractal structure of natural materials. Besides, the “cloud of internal stresses” term is introduced to describe the stress distribution in RVE. Its plastic flow is determined by the interaction of this cloud with the thick yield surface.
Thick yield surface and internal stress cloud concepts allow to determine the additional correlation between micromechanical models of polycrystals and phenomenological theory of plasticity.

## .1Introduction

Classical plastic flow models are not satisfying the modern requirements. There are two main directions in their development. The first, mathematical, is connected with the development of the general theory of rheological relationships; the second, physical, is connected with the development of the representative volume element (RVE) models which are based on specific ideas about deformation mechanisms. Within the framework of the first direction, the macroscopic experimental data is formalized; the second approach deals with the generalization of physical research results for the different scale levels that are involved in plastic deformation.

These approaches have advantages and disadvantages. In phenomenological theories, the advantages are the mathematical simplicity and laconic formulation of constitutive relations, which allows using them in practical calculations. However, the mathematical models of this class can not be considered as general, they are useful only for the description of limited range of deforming processes.

On the other hand, physical theories that are based on the description of real plastic deformation mechanisms are very explanatory and predictive, but they lack mathematical simplicity.

The above circumstances lead us to the necessity of compromise that can be found in determining the constitutive relations for RVE, which structure and parameters would bear the information about microlevel and mesolevel phenomena. In this synthesis direction, the fundamental role belongs to [1,2], etc.

In present work, we extend the definition of such a typically "macroscopic" concept as yield surface by means of including the additional information about material structure in it. "Thick yield surface" and "cloud of internal stresses" are introduced here to fulfill this need.

Yield surface is one of the fundamental concepts of mathematical theory of plasticity [3]. It represents a geometric image of the yield condition in the space of stresses. If a point mapping stressed condition of the material in this space resides inside the area limited by the yield surface, the material is deformed as elastic. In case when the indicated point hits on a surface, the plastic flow starts. The state mapped by points lying outside of the area limited by the yield surface is impossible. There are various interpretations concerning the form of yield surface and its evolution during the plastic deformations of material. In work [4] the explanation of sliding theory [5] is given within the framework of yield surface approach.

In several known works the yield surface is represented as the classical surface, i.e. two-dimensional object. Hypothesis of the present work is that the yield surface can be presented as thick, "foamed surface" with dimensionality exceeding two. By other words, perhaps, the yield surface is fractal, i.e. it belongs to the class of geometric objects with fractional dimensionality. Properties of such objects have been heavily studied [6]. Apparently, fractal structure of the yield surface is determined by the fractal structure of natural materials.

In many cases, the self-similarity of objects, or "scaling" as it is used by physicists, has already allowed to take into account their structure during the macroscopic description (see, for example, [6-8]). It inspires to have new attempts, as there are evidences (see, for example, [9]) about a fractal structure of metal alloys.

## .2Speculations about the "thick yield surface"

Let us represent RVE (characteristic linear dimension $l_{RVE}$) as a structure whose deformation is determined by the deformation of its interacting elements belonging to the scale level M1 (mesolevel 1 with a characteristic linear dimension $l_1$). The same way we will treat M1-level elements and so on, down to the simplest elements whose deformation can be described by certain mechanisms such as sliding, sliding with the change of volume, isotropic elements with the possibility to change the volume, and so on (See Figure 1). The elements that contain sub-elements will be called complex. At any level there can be both complex and simple elements (for instance, look at the M2-level on Figure 1).

To analyze the RVE behavior and the behavior of complex elements, let us introduce the following notions:

- Cloud of Internal Stresses (ISC) - area in the stress space that represents the stressed state of an element taking into account its inhomogeneity.

- Thick Loading Surface (TLS) - area in the stress space where, in which ISC experiences a plastic strain.

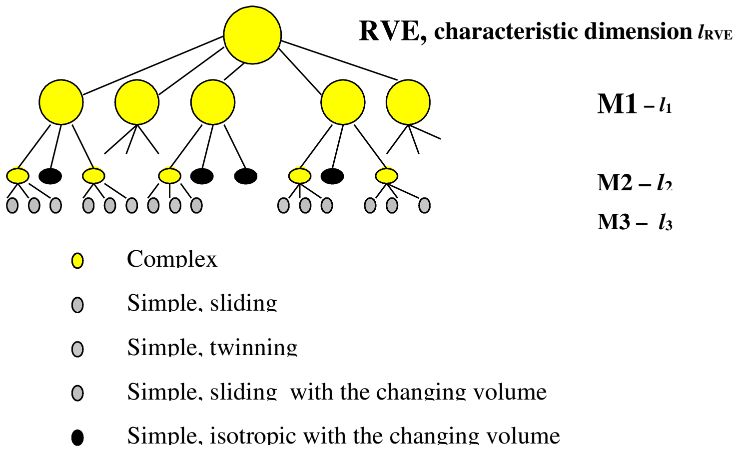


Figure 1. Hierarchical structure of the material.

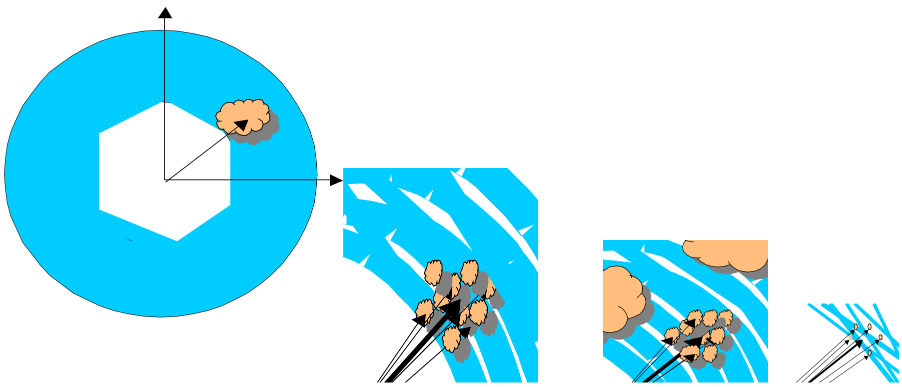

Figure 2. The structure of ISC and TLS.

The structure of ISC and TLS is shown on Figure 2 . You can clearly see that RVE recursively splits into smaller elements. Every split occurs by splitting a higher-level element into the corresponding elements of a lower level. This structure can be treated as a fractal in case of scaling.

Consider the following algorithm for constructing ISC and TLS.

Initially, loading surfaces of simple elements are obtained. They are represented by thick sheets of paper on the Figure 3d. Frontal and side views of the loading surfaces of simple elements are shown schematically. In this case an element has two sliding systems, one of which can experience densification or loosening, the other one cannot. By putting these sheets together, we get a book representing a complex element. The thickness of this book is $l_c$. You can see TLS and ISC of the element on the cover of the book. By combin-

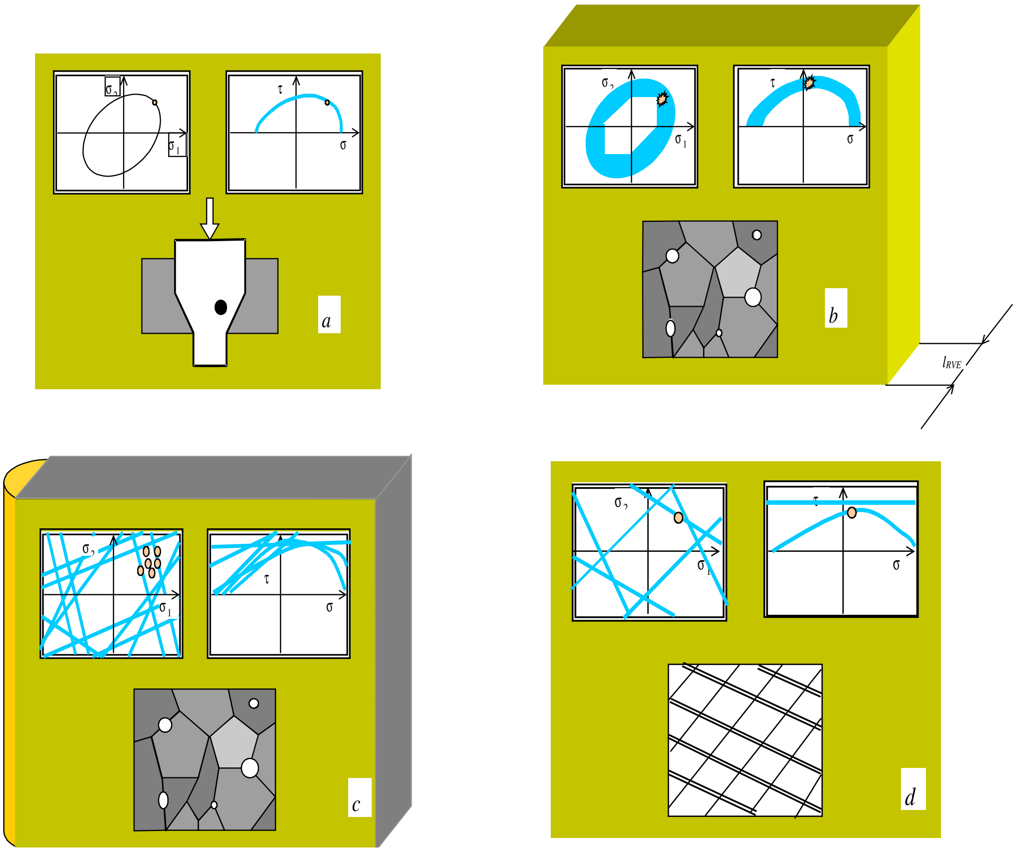


Figure 3 Algorithm of TLS and ISC construction.
*a* – RVE in classical understanding; *b* – RVE as a microinhomogeneous element;
*c* – complex element; *d* – simple element.

ing several books we can make a bigger one representing RVE; TLS and ISC of RVE are drawn on the cover of the bigger book. The last picture shows the classical loading surface of RVE, where RVE is considered to be homogeneous. That is why it appears as a sheet of paper with no thickness, and elements are shown as points in the deformed macro-body.

In order to investigate RVE deformation and build the corresponding TLS and ISC, we developed a cellular model of inhomogeneous material [10].

Our Cellular Model is based on the approach of a self-consistent field. In order to determine stress-strained state of RVE using self-consistent field approach, we use cellular automata technique. A uniform grid represents the area of interest. Each cell of the grid contains certain information. Time advances in discrete steps.

System laws are given by a set of rules, according to which any cell can determine its state at time *t*+1 based on its state and the state of its nearest neighbors at time t. It should be noted that these laws may have either dynamic or stochastic character. The latter allows investigating random processes. Besides, the model may be allowed to have memory, that is, to use the knowledge about the state of the system at previous time steps (*t*-1, *t*-2, etc).

A cellular automata allows to study the macrobehavior of the whole ensemble of cells depending on the local microscopic laws that determine the evolution of each cell and its interaction with the closest neighborhood.

We suggest to use a self-similar structure of cellular automata that allows to simulate the fractal nature of real materials.

We represented RVE as a system of interconnected elements with a multi-level hierarchical cellular structure. At each level, every element interacts with its nearest neighbors according to the laws of a self-consistent field. Due to its multi-level structure, the model can describe materials with inhomogeneous and fractal structure. We use complex cells that have a cubic lattice structure. Each such cell consists of 27 (3x3x3) smaller cells. In general, other spatial structures and a different number of components are possible. Please, note that the cells that I just described are not material cubes in the literal sense. Here they are just objects that are able to store certain information. In our case, the information may consist of parameters of stressed-strained conditions, indication of a membership in a certain class of objects, etc.

The cellular model allows us to have simple and complex elements. The deformation and rotation of complex elements are determined by their components, while the deformation and rotation of simple elements are determined by a certain mechanism (sliding, twinning, sliding with the volume change, and so on). Of course, to use the model, one needs to define the relations describing the interactions between elements and their nearest neighborhoods, and the relations describing the deformation of simple and complex elements. This relations are presented in [10].

The defining relationships for simple elements are based on loading surfaces corresponding to certain deformation mechanisms, and the law of associative flow.

In order to describe porous materials we introduce two elements that are able to change the volume during the deformation. Their governing relations are based on the model of structurally-inhomogeneous porous body. During the shift the material experiences two simultaneous processes: the healing of existing pores and the formation of new ones due to fracture of the material. The parameter $\alpha$ is the quantitative measure of the intensity of fracture. The more effective the relaxation mechanisms (other than mechanisms of pore-formation) are, the smaller the $\alpha$ is. In the extreme case, when there is no fracture of the material, $\alpha=0$.

In order to have a unique dependency of the speed of the element deformation on its strain, we introduce a non-linear viscosity instead of ideal plasticity.

## .3Polycrystal "thick yield surface" and its structure

Let us construct the yield surface of polycrystal in its classical understanding with the help of designed cellular model [10]. Figure 4a shows the results of calculations for the untextured polycrystal, figure 4b corresponds to the same calculation conditions for the textured material, allowed residual strain is $10^{-3}$. Figure 4a approaches Mises ellipse, figure 4b is very close to Tresca hexagon.

Trace of the yield surface calculated at the very small residual strain tolerance ($10^{-5}$) is shown on figure 4c. Calculation results have good correspondence with conclusions of Lipinski P. *et al.* [11], in which it is shown that the yield surface of polycrystal is similar to Tresca prism at the very small residual strain tolerance.

It is time to expand ideas about the yield surface. With this purpose, we will map the yield surfaces of separate elements in the space of stresses as if they were not connected to each other. Obviously, the yield surface of each sliding system will be mapped by a pair of parallel planes, determined by the equation

$$\left|\mathbf{m}^{(\alpha)}\mathbf{s}^{(\alpha)} : \Sigma\right| = \tau_c^{(\alpha)} . \tag{1}$$

The point corresponding to the stressed state of this system can be either between the indicated planes (in this case system is not active) or on them (in this case system is active).

The conjunction of yield planes of all sliding systems of simple units will eventually form the geometric object that we will name as "thick yield surface" of this unit. The quotation marks specify that this term is used not in classical understanding. Classical variant would be a yield surface representing the boundary of elastic areas intersection for all sliding systems of the considered unit [4].

"Thick yield surface" of *n*-level complex unit is a geometric object formed by association of "thick yield surfaces" of all (*n*+1)-level subitems contained in it. As a result of such construction in stress space, the self-similar geometric object is obtained. We will name it the "thick yield surface" of polycrystal. In the present work we propose a hypothesis that this object is fractal. Figure 5 shows the difference between classical and "thick" yield surfaces.

Different from the classical yield surface, the "thick yield surface" has an internal structure. Point mapping the stressed state of material representative volume can penetrate inside this "thick surface". Here it can "split" forming the set of points that map stressed states of the first level units. Each one of them can reside only within limits of its unit "thick yield surface", inside which this point, in turn, is "split" to the points corresponding to the stressed states of the second level units, etc. Finally, the stressed state of representative volume of polycrystal is mapped by a set of points *ISC*, which reside on yield planes of corresponding sliding systems of lowest-level units.

Obviously, diameter of *ISC* is a quantitative measure of internal microstresses in the material. On figure 6 the evolution of *ISC* under the loading of the same material in different directions within $\Sigma_{xx} - \Sigma_{yy}$ plane is shown (stressed states of units are given in 0-level unit co-ordinates). Calculation is executed using the cellular model of polycrystal at the conditions indicated above. The depth of a penetration inside the "thick yield surface" corresponds to the magnitude of residual strain 0.01.

Figure 6 shows that each penetration of the set *ISC* in "thick yield surface" results in distortion of this set. The beginning of plastic flow of material is connected with the first signs of such a distortion.

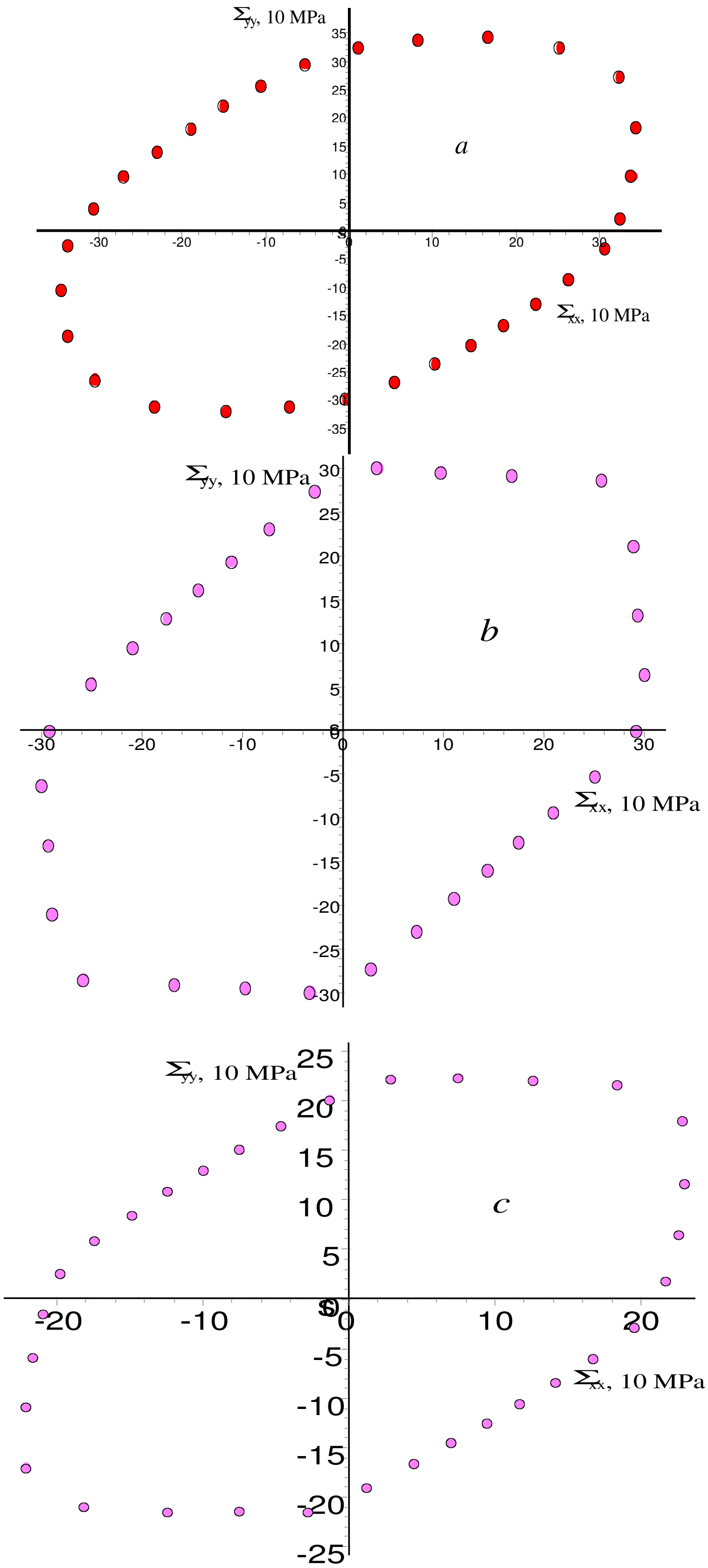


Figure 4. Calculated yield surface contours:
a – without texture; b – textured material; c – without texture at the very small residual strain tolerance ($10^{-5}$).

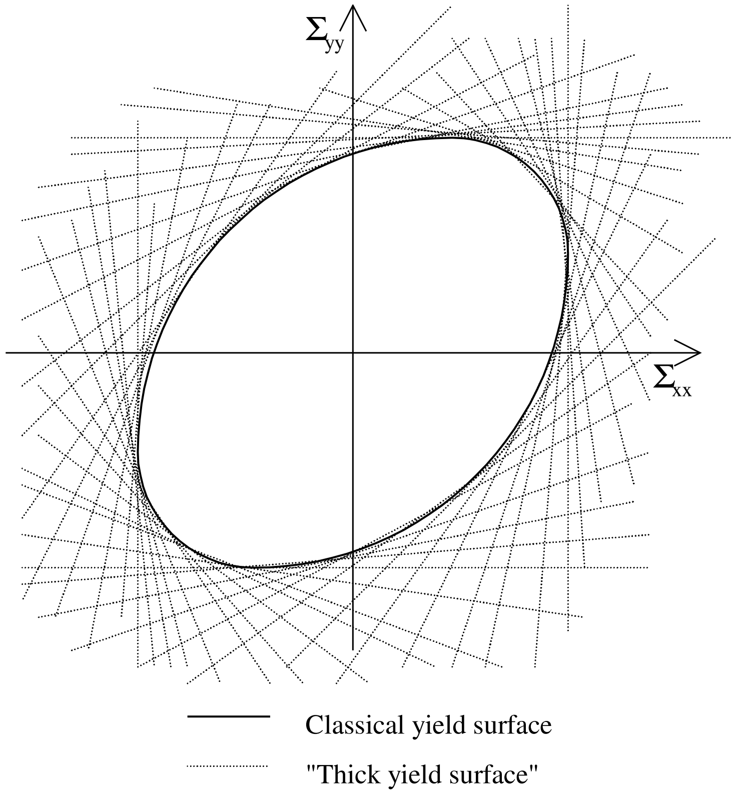


Figure 5. Difference between classical and "thick" yield surfaces.

Figure 7 shows the evolution of *ISC* at the complex two-stage loading going in opposite directions. It is clearly seen that the set ISC starts to distort at the smaller outer stresses during opposite sign loading, compared to the initial loading, which reveals the Baushinger's effect.

While the magnitude of penetration depth increases, the diameter of *ISC* grows, as each from the mapping points can move only along the yield plane, which corresponds to the growth of internal stresses.

The latter reaches maximum size and stops growing at the certain value of increasing strain. It means that at the rather deep penetration inside "thick yield surface" there should be rotation or fraction of yield planes, most considerably stretching the set *N*. Obviously, the first stated effect corresponds to the rotation of units, the second effect – to the partition of simple units, producing parts rotated one from another (fragmentation of grains). At the same time, the recess forms on the internal boundary of "thick yield surface".

Classical yield surface determines an increment of plastic deformation at each loading stage [3] by means of the associated flow law. It is easy to see that "thick yield surface", proposed by us, has similar properties. The difference is that in this case the macroscopic vector of plastic deformations increment of representative volume is obtained by summation of microstrain increments vectors of separate elements, which follows from the relationship (8). The increment of plastic deformations, which is stipulated by a sliding system, is determined by the vector that is orthogonal to the appropriate yield plane. The latter follows from (3), (5) and (31).

Velocity of polycrystal plastic deformation growth depends on the structure of the "thick yield surface" area, occupied by ISC, and on the direction of macrostress increment vector defining evolution of ISC.

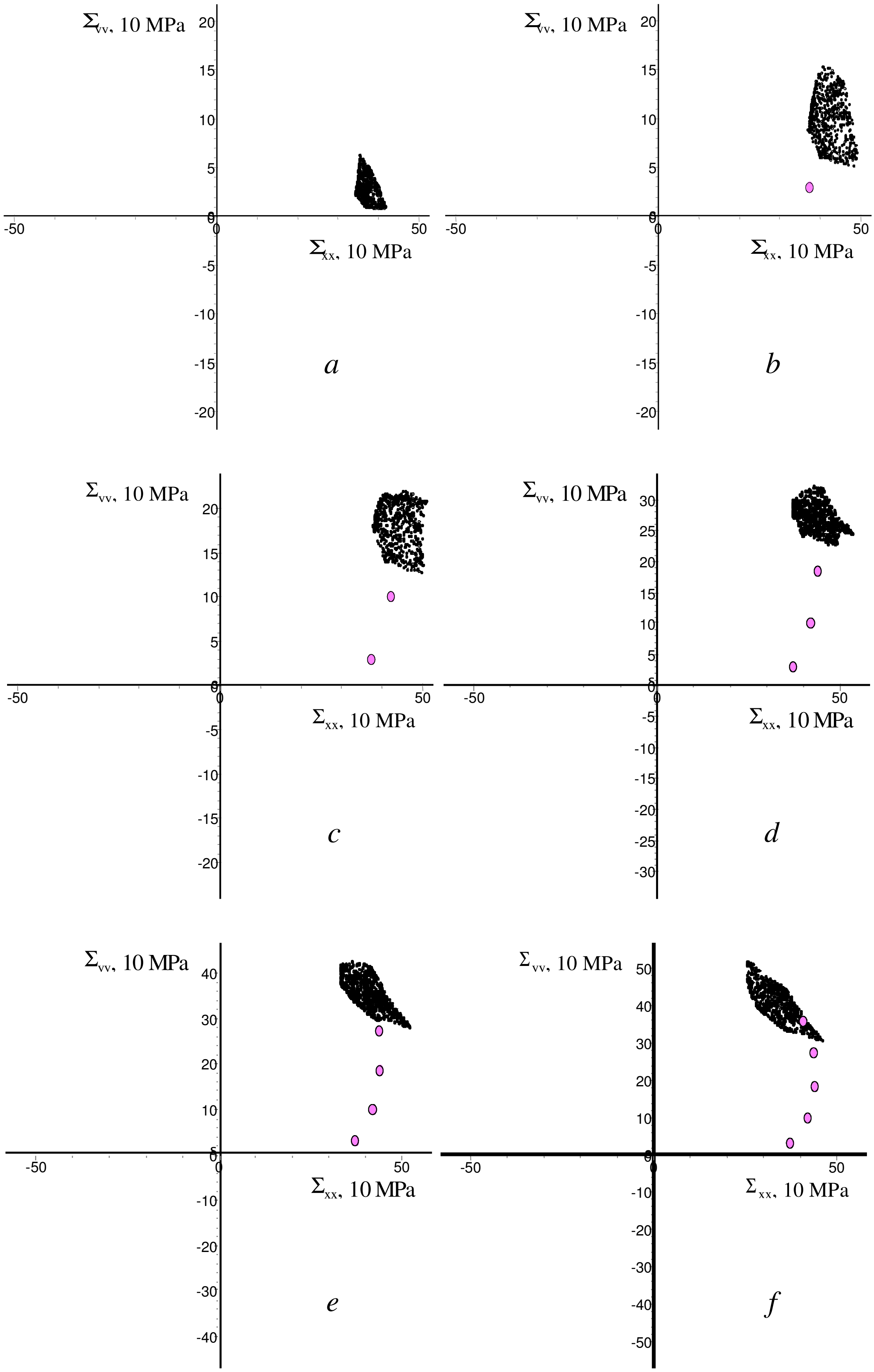


Figure 6. Evolution of ISC during the consequential penetration inside the "thick yield surface".

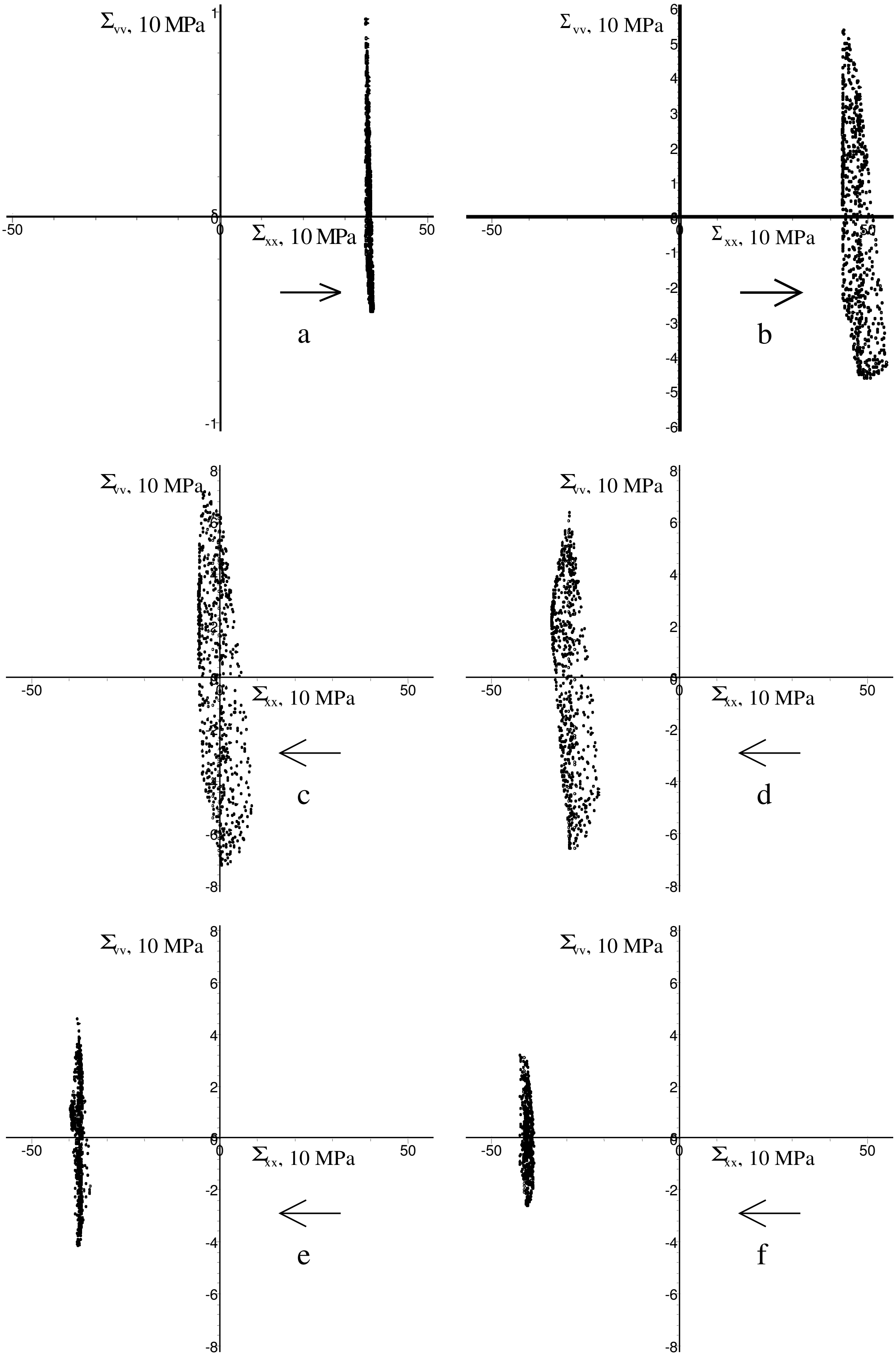


Figure 7. Evolution of ISC at the complex two-stage loading:
a – initiation of splitting at the direct loading; b – ISC at the end of direct loading;
c – ISC after unloading; d,e,f – distortion of ISC at the opposite sign loading.

In this connection, even though the magnitude of movement into the "thick yield surface" may be identical in specific cases, there may be different increments of plastic deformation in different parts of this surface and along the different directions of movement inside it. Here, the ideal plasticity (plasticity without hardening) corresponds to some limiting formations in a "thick yield surface". To reach those, there should be indefinitely large residual strain.

In the Table 1, we will summarize our experiments with a cellular model and describe several effects of plastic deformation using the language of TLS and ISC .

Table 1. Geometric Shapes of Selected Effects

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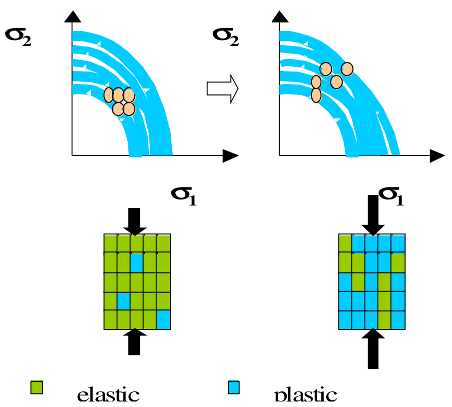
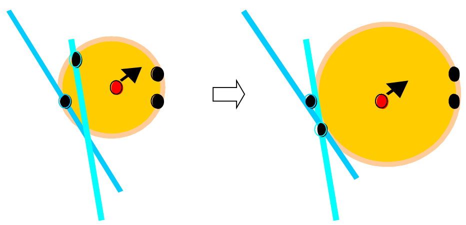
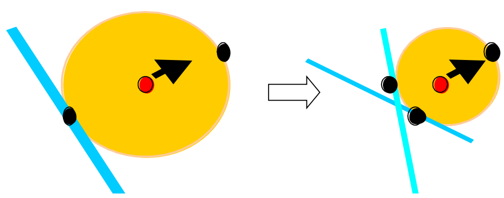
|---|---|
|   | **Elastically-plastic deformation and plastic flow of RVE**<br>As the load increases, more and more RVE elements become involved in the plastic deformation.<br>In addition, ISC spreads out, and even a larger number of its components start to penetrate their loading surfaces.<br>Plastic flow starts at the moment when the elastic cluster penetrating RVE, breaks. At this moment, a non-negligible fraction of ISC components are in their loading surfaces. |
|  | **Hardening caused by the increase of internal stress.**<br>Increase of the diameter of ISC under loading the material without grain fragmentation. |
|  | **Grain fragmentation under large deformations.**<br>Grain fragmentation caused by large internal microstress. In addition, the diameter of ISC decreases, and partial relaxation of internal microstress takes place. |

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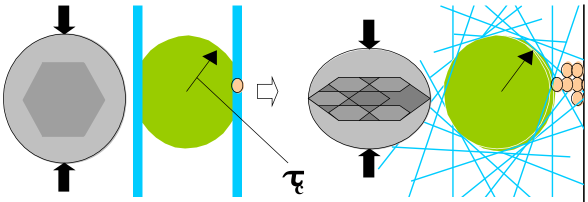
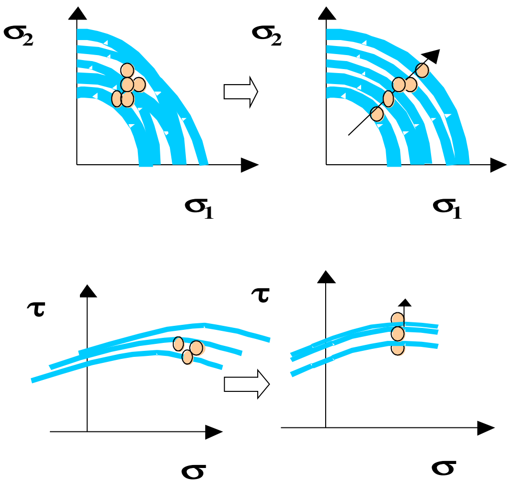
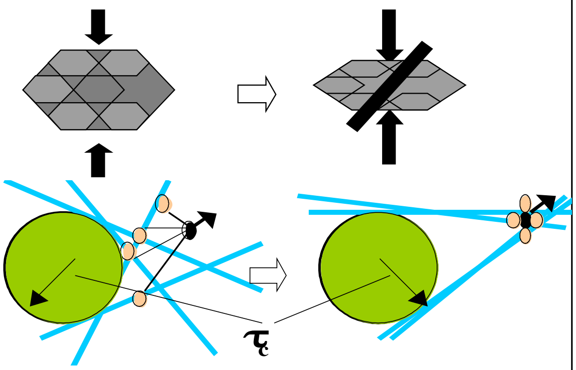
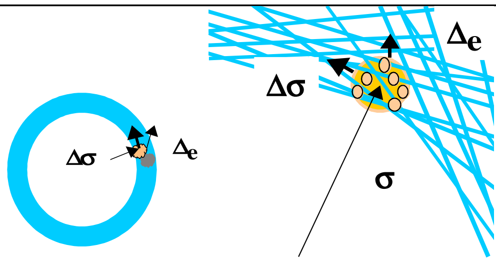
|---|---|
|  | **Hardening of RVE region caused by grain fragmentation under pressure.**<br>"Thin" loading surface transforms into "Thick" |
|  | **Ideal plasticity of microinhomogeneous material.**<br>Under large plastic deformations under high pressure, the following things happen:<br>• the material finally becomes isotropic;<br>• TLS gains the structure of concentrically ordered cylindrical hulls;<br>• ISC stretches along the radius of the cut of the hulls;<br>• In addition, a stationary porosity set in the RVE. |
|  | **Structure Formation. Loss of deformation stability with the formation of shear band**<br>Under grain turning the following things happen:<br>• There emerges a corner point in TLS;<br>• ISC collapses to this point. |
|  | **Non-proportional cyclic plasticity**<br>Deformation under loading along TLS [12]. |

**.4Conclusions**

To summarize everything that was stated about the thick yield surface, it is possible to conclude that, due to a number of reasons, it is an interesting object, especially while considered together with the cloud of internal stresses. This object should not be reduced just to a new label. Several reasons from the above-mentioned are given below.

- Thick yield surface allows distinguishing between the hardening mechanisms connected with internal stresses and those connected with modifications in material structure. First ones are revealed as modifications of "cloud of internal stresses" and have no effect on thick yield surface; second ones change this surface.
- Thick yield surface allows to give a new geometric image to effects caused by internal stresses and by disproportionate loading and unloading. For example, the Baushinger's effect is explained not as a migration of the whole yield surface, but rather as extension of the internal stresses cloud. Plastic flow during the neutral additional loading can be connected not with formation of the angular point on a smooth yield surface. Here, the center of internal stresses cloud (point mapping the stressed state of RVE as whole) is moving inside the body of thick yield surface.
- Numerical experiments show that the size and the form of internal stresses cloud depend on the trajectory and the length of path followed by the center of this cloud along the trajectory of its movement inside "thick yield surface". In turn, the geometry of internal stresses cloud and local structure of thick yield surface determine RVE deformation velocity. All this encourages to think that the geometric properties of the proposed objects can be used as parameters in constitutive relationships for RVE.

Concept of the thick yield surface allows obtaining the additional correlation between the micromechanical models of polycrystals and the phenomenological theory of plasticity. Evidently, the thick yield surface is fractal. Structure of this "surface" as well as other properties of deformable polycrystal can be investigated with the help of the developed cellular model.

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