Adaptive Isomap

11th November 2003

1 Motivation and Goals

In the proof that isomap is sufficient to extract an isometrically embedded manifold, success is dependent upon having a number of samples related to the worst radius of curvature over the manifold. This is clearly excessive when most of the embedded manifold is flat. Can we construct a manifold extraction algorithm with the property that the number of samples required is not worst case over all curvatures?

2 Definitions and Theory thoughts

Let $d_M(x, y)$ be the distance between point x and point y in the manifold M. Let $d(x, y)$ be the euclidean distance between x and y in the embedding space.

Let

$$D(x, \epsilon) = \max\{d : \forall y \text{ with } d(x, y) \leq d \;\; d_M(x, y) \leq d(x, y)(1 + \epsilon)\}$$

Intuitively, $D(x, \epsilon)$ is the maximal distance in the embedding space such that the distances in the embedding space and in the manifold agree up to additive error ϵ.

A quick property:

If $d(x, y) \leq D(x, \epsilon)$ then $0 \leq D(y, \epsilon/2) \leq 2D(x, \epsilon)$

Suppose we have a set of points S which is ϵ-distortion n-dense: $\forall x \in M \exists y \in S : d(x, y) < \frac{D(x,\epsilon)}{n}$

Can we reconstruct the manifold?

Suppose we knew the value of $D(x, \epsilon)$ for every $x \in S$. If we construct a neighborhood graph using $D(x, \epsilon)$-nearest neighbor, the multiplicative distortion of any geodesic estimate is only $1 + \epsilon$. Furthermore, the distortion due to path lengthening in the graph is controlled as n grows.

3 An algorithm

All of the above is a motivation which isn't easy (but may not be impossible) to turn into an algorithm. The biggest difficulty is that we do not know $D(x, \epsilon)$ at run time. However, this may be an estimatable property.

Let's suppose that we know the underlying dimension d as well as the point set. Let's also assume that the branch seperation is large. Then, given the set S, we could estimate $D(x, \epsilon)$ as the radius such that the dimensionality (approximately) spanned by the points grows to be larger than d.

This would imply an algorithm similar to the current isomap, except that the neighborhood grows until a "local flatness" test is failed. Presumably, the local neighborhood could be sparsified by eliminating vectors with a small dot product in order to retain tractability.