A Little History:

AdaBoost came out of the PAC Learning community

- PAC Learning model developed by Valiant ("A theory of the Learnable" 1984)
- Kearns & Valiant (1988, 1994) posed the question of whether a "weak" PAC Learning algorithm, e.g., one that performs slightly better than a random guess, could be used to construct a strong PAC Learning algorithm, i.e., one that performs arbitrarily well (yet still poly-time & obeys other technical conditions).

Kearns & Servedio were Valiant's students

- Freund & Schapire's example: betting strategies for horse racing:

  An expert gambler may not be able to explain his/her betting strategy, but when presented w/ data for a specific set of races, s/he can give us a "rule of thumb"

  - bet on the horse that has won the most recently
  - " " " " w/ the most favored odds

  ⋮

  Rules of thumb are not very accurate, but a little better than a random guess.

  Boosting algorithms combine these rules into a single, highly accurate prediction rule.

- Schapire (1989) - showed that the answer to Kearns & Valiant's question was "yes". Proof by construction of the 1st boosting algorithm. (Wasn't so practical though!)
- Between '89-'95 a few other boosting algs were designed but none were very practical.
- Freund & Schapire (1995) "A decision-theoretic generalization of online learning & an application to boosting (1997) - introduced AdaBoost.
- Schapire & Singer (1999) - extended AdaBoost to more general rules of thumb.

Let's derive AdaBoost (But not in the way F+S did it. Turns out AdaBoost is stagewise optimization, discovered by at least 5 groups. We'll do it that way.)

Standard Classification Task:

training set : $\{(x_i, y_i)\}_{i=1 \ldots m}$ $x \in \mathcal{X}$, $y \in \{-1, 1\}$ chosen randomly from unknown prob. dist'n.

Want to find $f: \mathcal{X} \Rightarrow \mathbb{R}$ such that $\text{sign}(f(x))$ agrees with y as much as possible, $f \in \mathcal{F}$.

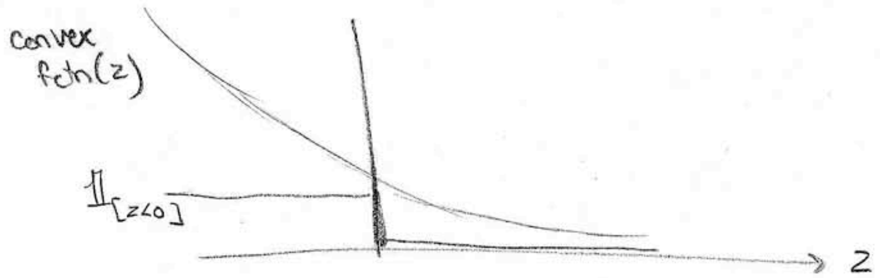
(We hope that if our chosen f performs well on the training set, it will perform well on the whole prob. dist'n.)

misclassification error of f w.r.t. training set $=$ # of times $y_i \neq \text{sign}(f(x_i))$

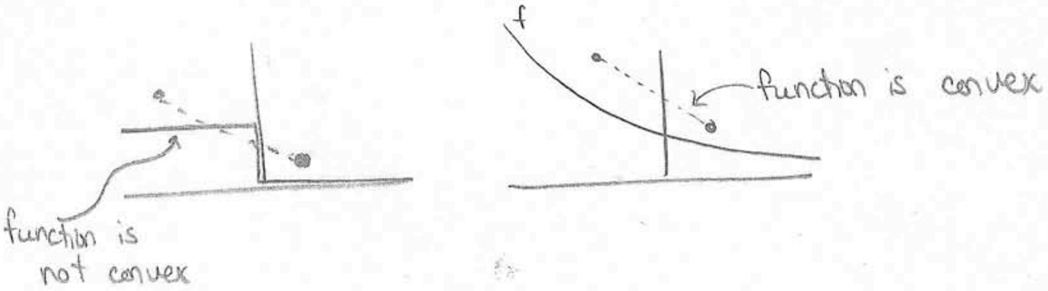
$=$ # of times $y_i f(x_i) < 0$

$= \sum_{i=1}^{m} \mathbb{1}_{[y_i f(x_i) < 0]}$ ← indicator function is 1 if condition holds, 0 otherwise

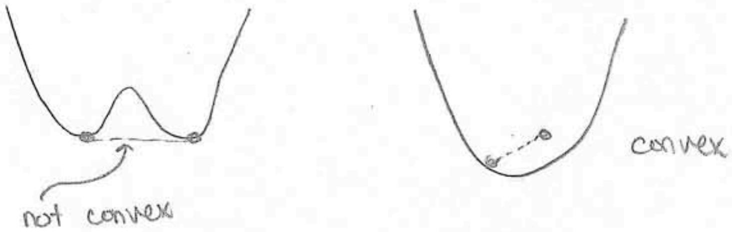
We want misclassification error to be small, i.e, would like to choose f to minimize misclassification error. The problem is that in terms of practical optimization, minimizing this quantity is very difficult. It would be much easier if the misclassification error were convex.



Aside: A function is convex if and only if the set of points lying on or above the graph is a convex set. Pick any 2 points above the graph of f. If f is convex, the whole line of points connecting the 2 points is also above f.



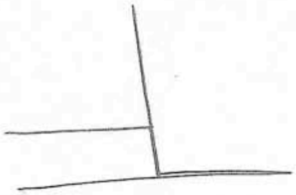
For convex functions, all <u>local minima</u> are <u>global minima</u>.



Sums of convex fctns are convex, and other nice properties.

Let us perform a trick, namely to use:

$$\mathbb{1}_{[z<0]} \leq e^{-z} \qquad (*)$$



Thus,

$$\text{misclassification error of } f \text{ w.r.t. training set} = \sum_{i=1}^{m} \mathbb{1}_{[y_i f(x_i) < 0]}$$

$$\leq \sum_{i=1}^{m} e^{-y_i f(x_i)} =: L(f)$$

We hope that choosing an f to yield small values of L(f) will yield small values of the misclassification error.

Need to choose the form of f. For AdaBoost, f is a linear combination of "weak classifiers" or "rules of thumb".

$$f(x) = \sum_{j=1}^{n} \lambda_j h_j(x) \quad \text{where } h_j : \mathcal{X} \to \{-1, 1\}$$

AdaBoost's Objective Function: $L(\lambda) = \sum_{i=1}^{m} e^{-\sum_{j=1}^{n} \lambda_j y_i h_j(x_i)}$

Want to minimize this w.r.t. λ.

About the weak classifiers $\{h_j\}_{j=1..n}$

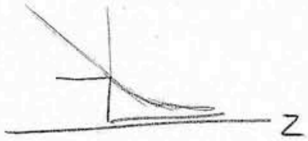
- AdaBoost can be used in 2 ways, either to do most of the work, or as a "wrapper" to increase the accuracy of an already accurate base learning algorithm (e.g., boosted neural networks). In the second case, the h_j's are classifiers that come from the base learning algorithm.
- n can be huuge... or even infinite.

We assume the h_j's are given to us, so we just need to find λ.

Aside: Derivation of Logistic Regression's Objective:

If instead of ⊛ we had used

$$\mathbb{1}_{[z<0]} \leq \log(1+e^{-z})$$



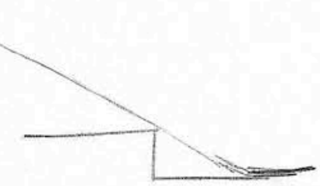
then we would derive the objective for the classification algorithm called Logistic Regression.

$$L_{LogReg}(\lambda) = \sum_{i=1}^{m} \log\left(1 + e^{-\sum_{j=1}^{n} \lambda_j y_i h_j(x_i)}\right)$$

Aside: Part of SVM's objective:

If instead of ⊛ we had used

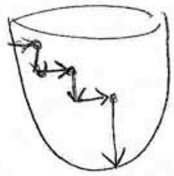
$$\mathbb{1}_{[z<0]} \leq \begin{cases} 1-z & z \leq 1 \\ 0 & z \geq 1 \end{cases}$$



then we would have derived part of the objective for SVM's. (More on this another day.)

Back to AdaBoost!

Since $L(\lambda)$ is convex in λ, we can use simple techniques to minimize $L(\lambda)$ w.r.t. λ in $\mathbb{R}^n$. We'll use "coordinate descent":



for $t = 1 .. T$

Step 1: Find the steepest "direction" j_t (ie. choose a weak classifier)

Step 2: move along that direction until $L(\lambda)$ is minimized

(ie choose α to minimize $L(\lambda + \alpha e_{j_t})$ where $e_{j_t} = \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \vdots \\ \\ \\ j_t \\ \\ \vdots \end{array}$

Formally,

Objective $L(\lambda) = \sum_{i=1}^{m} e^{-\sum_j \lambda_j y_i h_j(x_i)}$

Directional Derivative
$$\left.\frac{dL(\lambda + \alpha e_{\tilde{j}})}{d\alpha}\right|_{\alpha=0} = \sum_{i=1}^{m} -y_i h_{\tilde{j}}(x_i)\, e^{-\sum_j \left(\lambda_j y_i h_j(x_i) + \cancel{\alpha y_i h_{\tilde{j}}(x_i)}\right)}\Bigg|_{\alpha=0}$$

Step 1:
$$j_t = \underset{\tilde{j}}{\operatorname{argmax}} \left.\frac{-dL(\lambda + \alpha e_{\tilde{j}})}{d\alpha}\right|_{\alpha=0} = \underset{\tilde{j}}{\operatorname{argmax}} \sum_{i=1}^{m} + y_i h_{\tilde{j}}(x_i)\, e^{-\sum_j (\lambda_j y_i h_j(x_i))}$$

$$= \underset{\tilde{j}}{\operatorname{argmax}} \sum_{i=1}^{m} + y_i h_{\tilde{j}}(x_i)\, d_i \qquad \text{where } d_i = \frac{e^{-\sum_j \lambda_j y_i h_j(x_i)}}{Z \leftarrow \text{normalizer}}$$

Step 2:
$$0 = \frac{-dL(\lambda + \alpha e_j)}{d\alpha} \propto \sum_{i=1}^{m} + y_i h_{j_t}(x_i)\, \frac{e^{-\sum_j \left(\lambda_j y_i h_j(x_i) + \alpha y_i h_{j_t}(x_i)\right)}}{Z}$$

$$= \sum_{i:\, y_i h_{j_t}(x_i) = 1} +1\, d_i e^{-\alpha} + \sum_{i:\, y_i h_{j_t}(x_i) = -1} -1\, d_i e^{\alpha}$$

$$= +(1 - d_-) e^{-\alpha} - d_- e^{\alpha} \qquad \text{where } d_- = \sum_{i:\, y_i h_{j_t}(x_i) = -1} d_i$$

$$(1 - d_-) e^{-\alpha} = d_- e^{\alpha}$$

$$\frac{1 - d_-}{d_-} = e^{2\alpha} \rightsquigarrow \alpha = \frac{1}{2} \ln \frac{1 - d_-}{d_-}$$

Pseudocode for AdaBoost:

Given $\{(x_i, y_i)\}_{i=1..m}$, $\{h_j\}_{j=1..n}$, T, $\lambda_{1j} = 0 \ \forall j$

For $t = 1..T$, $d_{1,i} = \frac{1}{m} \ \forall i$

$$j_t = \underset{\tilde{j}}{\operatorname{argmax}} \sum_{i=1}^{m} -y_i h_{\tilde{j}}(x_i) d_i$$ "train weak learner" using dist'n d

$$d_{t-} = \sum_{i: y_i h_{j_t}(x_i) = -1} d_{t,i}$$ error of weak classifier

$$\alpha_t = \frac{1}{2} \ln \left(\frac{1 - d_{t-}}{d_{t-}} \right)$$

$$\lambda_{t+1} = \lambda_t + \alpha_t e_{j_t}$$ add weak classifier's contribution

$$d_{t+1,i} = d_{t,i} \cdot \left\{ \begin{array}{ll} e^{-\alpha_t} & \text{if } h_{j_t}(x_i) = y_i \\ e^{\alpha_t} & \text{if } h_{j_t}(x_i) \neq y_i \end{array} \right\} \Big/ Z_t$$ write weights for next round in terms of weights for this round

end

To evaluate f,

$$f(x) = \sum_j \lambda_{T_j} h_j(x).$$

Notes: This is not the usual way AdaBoost is introduced or derived. Usually one thinks of the d_{ti}'s as fundamental, viewing them as "weights" on each training example. (Here, d_{ti}'s fall naturally out of the derivation.)

At each t, give more credit to classifiers that did well w.r.t. the weighted training examples.